Algebra III

Mid-Term Examination.

September 2004

INSTRUCTIONS. Answer any five questions. $\mathbb{Q}$ denotes the field of rational numbers. All fields are assumed to be of characteristic zero.

1. (a) Let L be an algebraic extension of a field F. Prove that every subring of L which contains F is a field. (5 Marks)

   (b) Let $k \subset F \subset L$ be a tower of fields. Prove that $[L:k] = [L:F]\,[F:k]$. (5 Marks)

2. (a) Let F be a field and let f, g be two polynomials in $F[X]$. If L is an extension of F, prove that the g.c.d of f and g in $F[X]$ equals that in $L[X]$. (4 Marks)

   (b) Let F be a finite extension of $\mathbb{Q}$. Show that there exists an element $\theta \in F$ such that $F = \mathbb{Q}(\theta)$. (6 Marks)

3. (a) Let L be a subfield of the field of complex numbers such that its degree over $\mathbb{Q}$ is finite. If -1 is a sum of squares in L, prove that $[L:\mathbb{Q}]$ is an even number. (4 Marks)

   (b) Let $\phi : F \to K$ be a field isomorphism and $f \in F[X]$ be a polynomial. Let L and L' be the splitting fields of f and $\phi(f)$ over F and K respectively. How many field isomorphisms $\phi_i : L \to L'$ exist such that ϕ_i restricted to F equals ϕ ? Justify your answer. (6 Marks)

4. Let L be a degree 6 extension of $\mathbb{Q}$.

   (a) Show that L contains at most one degree 2 extension of $\mathbb{Q}$. (2 Marks)

   (b) If L contains more than one degree 3 extension of $\mathbb{Q}$, then show that L is a Galois extension of $\mathbb{Q}$. (4 Marks)

   (c) Give an example of a degree 6 Galois extension of $\mathbb{Q}$ which contains exactly one degree 3 extension of $\mathbb{Q}$. Give an example of a non-Galois extension of degree 6 over $\mathbb{Q}$. (4 Marks)

5. (a) Let L be a Galois extension of $\mathbb{Q}$ with Galois group S_n, the permutation group on n symbols. Prove or disprove: L is a splitting field of a degree n irreducible polynoimal in $\mathbb{Q}[X]$. (5 Marks)

   (b) Determine the Galois group of $(X^2-2)(X^3-2)$ over $\mathbb{Q}$. (5 Marks)

6. (a) Let K be a Galois extension of $\mathbb{Q}$ with Galois group G. Let $F \subset K$ be a field corresponding to the subgroup H of G. Show that $N(H)/H$ is the group of automorphisms of F (here $N(H)$ denotes the normaliser of H in G.) (5 Marks)

   (b) Let K be a Galois extension of $\mathbb{Q}$. Let $f \in \mathbb{Q}[X]$ be an irreducible polynomial such that $f = g.h$ in $K[X]$ with g, h monic irreducible. Show that there exists an automorphism σ of K such that $\sigma(g) = h$. Give an example when this conclusion is not valid if K is not a Galois extension of $\mathbb{Q}$. (5 Marks)

7. (a) Find the splitting field of the polynomial $X^4 - 4X^2 - 1$ over $\mathbb{Q}(\sqrt{5})$. (3 Marks)

   (b) Find the Galois group of the above polynomial over $\mathbb{Q}$ and over $\mathbb{Q}(\sqrt{5})$. (7 Marks)